Closure of two-dimensional turbulence: The role of pressure gradients

G. Boffetta,[1] M. Cencini,[2,3] and J. Davoudi[4,5]
[1]*Dipartimento di Fisica Generale and INFM, Università di Torino, Via Pietro Giuria 1, I-10125 Torino, Italy*
[2]*Dipartimento di Fisica, and INFM, Università di Roma "la Sapienza," Piazzale Aldo Moro 5, 00185 Roma, Italy*
[3]*Observatoire de la Côte d'Azur, Nice Boulevard de l'Observatoire, Boîte Postal 4229 F-06304 NICE Cedex 4 France*
[4]*Max-Planck-Institute für Physik Komplexer Systeme, Nöthnitzer Straße 38, 01187 Dresden, Germany*
[5]*Fachbereich Physik, Philipps Universität, Renthof 6, 35032 Marburg, Germany*
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Inverse energy cascade regime of two-dimensional turbulence is investigated by means of high resolution numerical simulations. Numerical computations of conditional averages of transverse pressure gradient increments are found to be compatible with a recently proposed self-consistent Gaussian model. An analogous low-order closure model for the longitudinal pressure gradient is proposed and its validity is numerically examined. In this case numerical evidence for the presence of higher-order terms in the closure is found. The fundamental role of conditional statistics between longitudinal and transverse components is highlighted.

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The existence of two simultaneous inertial ranges in two-dimensional turbulence, as a consequence of coupled energy and enstrophy conservation, is one of the most important phenomena in statistical fluid mechanics [1]. At variance with three-dimensional turbulence, the energy injected into the system at scale ℓ_f flows toward the large scales, while the enstrophy cascades down on the small scales. Because of the inverse energy cascade, the Navier-Stokes equations,

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i p + \nu \partial^2 u_i + f_i , \qquad (1)$$

which rule the evolution of an incompressible ($\partial_i u_i = 0$) velocity field, cannot reach a steady state unless an energy sink at large scales is added. Alternatively, one can consider an ensemble of solutions of Eq. (1) with a fixed energy value below the condensation level [2], i.e., with an integral scale $L(t)$ (growing in time as $t^{3/2}$) still much smaller than the system size. Because of the scaling of the characteristic times, the small scales (inertial range) in the system $\ell_f \ll r \ll L$ can be considered in a stationary state. One of the most challenging problems is to understand the statistics of velocity fluctuations $\Delta\mathbf{u}(\mathbf{r}) = \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$ [3]. In homogeneous and isotropic turbulences it amounts to study the *joint* probability density function (PDF) $P(U,V,r)$ of longitudinal U and transverse V velocity differences, where $\Delta\mathbf{u} = U\hat{\mathbf{x}} + V\hat{\mathbf{y}}$ and $\hat{\mathbf{x}} = \mathbf{r}/r$. Recently experimental [4] and numerical [5] investigations in two dimensions have shown that the probability distribution of the *pure* longitudinal $P(U,r)$ and transversal $P(V,r)$ velocity differences at inertial scales display a close-to-Gaussian statistics with undetectable intermittency corrections to structure function exponents. Although the establishment of normal scaling in all inverse cascades seem to be generic [6], nevertheless the Gaussianity of the statistics in inverse cascade of the forced two-dimensional turbulence remained to be understood. From Eq. (1), a set of equations for generic mixed structure functions, i.e., $S_{n,m}(r) \equiv \langle U^n V^m \rangle = A_{n,m} r^{\xi_{n,m}}$ have been obtained [7,8]. In Ref. [8] those equations are elaborated from the joint PDF equation. Unfortunately, the PDF equation is not closed, resembling the well-known closure problem in turbulence. In the inverse energy cascade regime, dissipative contributions can be neglected, so that the remaining unclosed terms are the longitudinal and transversal pressure gradient increments. Recently Yakhot [8,9] suggested a self-consistent model for the pressure gradient increments and succeeded to obtain a Gaussian distribution for the transverse PDF, $P(V,r)$. Although the experimental [4] and numerical [5] observations support the Gaussian result of the effective low-order model, nevertheless a direct numerical computation of the pressure gradient increment contribution is still lacking.

The main aim of this work is to compare the numerical evaluation [10] of transverse and longitudinal components of pressure gradient increments with the theoretical predictions of a recently introduced closure scheme. We emphasize on the importance of velocity mixed conditional averages, such as $\langle U|V,r\rangle$ and $\langle V^2|U,r\rangle$ generally arising in the pure longitudinal or transversal PDF equations. To our surprise the existence of such objects has been neglected in all the previous theoretical modelings. As an essential step for the description of pure velocity statistics we numerically evaluate the behavior of these objects for which some effective models are proposed. Such an investigation provides a direct check of the closure model.

By standard statistical tools [8,11], starting from the Navier-Stokes equations (1), it is possible to derive the following exact PDF equation for joint transversal and longitudinal velocity increments:

$$\left[\partial_r U + \frac{U}{r} - \frac{1}{r}\partial_V UV + \frac{1}{r}\partial_U V^2\right] P(U,V,r) = [\varepsilon(\partial_U^2 + \partial_V^2) + \partial_U \mathcal{P}_{x,u} + \partial_V \mathcal{P}_{y,v}] P(U,V,r), \qquad (2)$$

where $\varepsilon \equiv \langle f_i u_i \rangle$ is the rate of energy input and the conditional transversal, $\mathcal{P}_{y,v} \equiv \langle \Delta \partial_y p | U,V,r\rangle$, and longitudinal, $\mathcal{P}_{x,u} \equiv \langle \Delta \partial_x p | U,V,r\rangle$, pressure gradient increments are the only unclosed terms. In pure longitudinal and transversal PDF equations other unknown quantities play role. Indeed, by integrating Eq. (2) over U or V the terms $\langle U|V,r\rangle = \int_{-\infty}^{+\infty} UP(U|V,r)dU$ and $\langle V^2|U,r\rangle = \int_{-\infty}^{+\infty} V^2 P(V|U,r)dV$

appear in the pure transverse or longitudinal PDF equations, respectively, pinpointing the statistical dependence between longitudinal and transversal components. Let us start with the transversal one, for which the knowledge of $\langle\Delta\partial_y p|V,r\rangle=\int_{-\infty}^{+\infty}\mathcal{P}_{y,v}P(U|V,r)dU$ and of $\langle U|V,r\rangle$ is sufficient to close the equation. Following the recently proposed closure [8], we *assume* a second-order expansion for the transverse pressure gradient increments $\mathcal{P}_{y,v}$ in terms of *local* velocity increments U and V. Even if the locality assumption is not based on rigorous grounds, there are some arguments in support of its plausibility [9]. Once locality is accepted, keeping only second-order terms is motivated from the fact that for Gaussian fields only quadratic combinations of U and V appear [12]. Some physical constraints simplify the expansion even further [13,14], ending with Yakhot ansatz [8]

$$\langle\Delta\partial_y p|U,V,r\rangle=-h\frac{UV}{r}-b(\varepsilon r)^{1/3}\frac{V}{r}. \tag{3}$$

To directly check the closure one has to compute quantities like $\mathcal{P}_{y,v}$. However, to be more quantitative, here we numerically compute $\langle\Delta\partial_y p|V,r\rangle$ and $\langle\Delta\partial_y p|U,r\rangle$ for which we have a better statistics. For symmetry reasons $\langle\Delta\partial_y p|U,r\rangle=0$ as confirmed by simulations, and we are left with the analysis of the term $\langle\Delta\partial_y p|V,r\rangle$. We start by writing the quantities of interest in a scale-invariant form. For a scale-invariant solution for the PDF equation, i.e., $P(V,r)=P[V/(\varepsilon r)^{1/3}]\equiv P(X)$, is sufficient to require scale invariance of $\langle U|V,r\rangle$ and $\langle\Delta\partial_y p|V,r\rangle$. We thus define $\langle U|V,r\rangle=(\varepsilon r)^{1/3}F(X)$ and $\langle\Delta\partial_y p|V,r\rangle=[(\varepsilon r)^{2/3}/r]G(X)$. The major challenge now is to determine the functional form of $G(X)$ and $F(X)$. Taking into account the symmetries of Eq. (1), we assume for $\langle U|V,r\rangle$ an even polynomial expansion in V. Invoking the homogeneity, $\overline{\langle U|V,r\rangle}=0$, leads to the low order expansion:

$$F(X)=C_2(-A_{0,2}+X^2), \tag{4}$$

meaning that positive (negative) longitudinal velocity increments correspond to large (small) transverse velocity increments. Furthermore, by integrating Eq. (3) over U one obtains $G(X)=-hXF(X)-bX$. Apparently this is a two-parameter expansion, however, the constraint $\overline{V\mathcal{P}_{y,v}}=0$ [7,8] implies $h\overline{X^2F(X)}=-b\overline{X^2}$. Since $\overline{X^2F(X)}=A_{1,2}=1/2$, one ends up with the relation $hA_{1,2}=-bA_{0,2}$. The important fact is that this expansion is consistent with Gaussianity of transverse fluctuations and also gives a reasonable account for pressure contributions in the structure function equations [7,15,16]. Indeed plugging the expansion for F and G in the dimensionless transverse PDF equation, one obtains the Gaussian result $P(X)=\exp(-X^2/2A_{0,2})$ [8,16], which is consistent with simulations and experiments [4,5]. Since positivity and finiteness of the PDF fixes the constant $C_2=1/(4A_{0,2}^2)$ and $h=4/3$, therefore $A_{2,0}=3/5A_{0,2}$ is the only free parameter of the theory [8,16]. Therefore, within second-order approximation one has

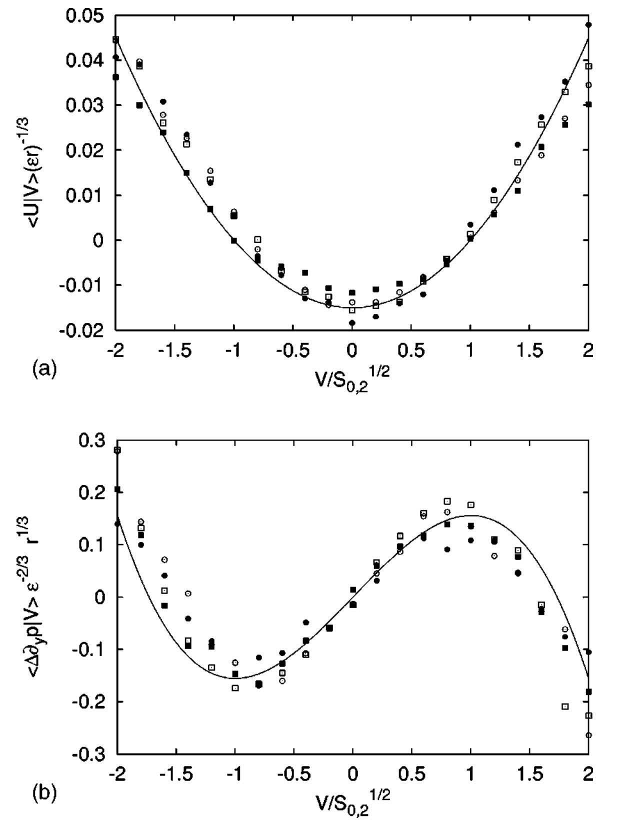

FIG. 1. (a) $\langle U|V,r\rangle$ and (b) $\langle\Delta\partial_y p|V,r\rangle$ computed at $r=0.025$ (boxes) and $r=0.037$ (circles). Empty symbols refer to the Gaussian forcing and full ones to the one restricted in a band of wave numbers. The full lines represent predictions (5) and (6) with $A_{2,0}=11$.

$$\langle U|V,r\rangle=\frac{(\varepsilon r)^{1/3}}{4A_{0,2}^2}(-A_{0,2}+X^2), \tag{5}$$

$$\langle\Delta\partial_y p|V,r\rangle=\frac{(\varepsilon r)^{2/3}}{r}\left(\frac{X}{A_{0,2}}-\frac{X^3}{3A_{0,2}^2}\right), \tag{6}$$

which up to about two standard deviations agree remarkably well with the numerical data (see Fig. 1). Moreover, using Eq. (5) as a fitting function, we obtained $A_{2,0}=11\pm1$, which is close, within the statistical errors, with the value obtained in previous experimental [4,17] numerical [2,5,18]), and analytical [19] studies. We remark that the good agreement of direct numerical simulation (DNS) data with Eq. (6) provides a first evidence (even if numeric) for the plausibility of the locality assumption. However, one can verify that assuming higher-order polynomials for $F(X)$ can result in nonzero higher-order terms in $G(X)$. Indeed plugging the Gaussian result in the equation for $P(X)$, for any order consistent with Gaussianity, $G(X)$ is expressible as a functional of $F(X)$. So we obtain [16]

$$G(X)=-\frac{X}{A_{0,2}}-\frac{4}{3}\left(XF(X)+e^{(X^2/2A_{0,2})}\times\int^X F(X')e^{-(X'^2/2A_{0,2})}dX'\right). \tag{7}$$

Plugging the self-consistent low-order model (4) in Eq. (7) will reduce the proposal of Yakhot. Equation (7) provides a way to generalize the model $G(X)$, in a self-consistent way, to higher-order polynomials. It is evident that higher-order terms in $F(X)$ can lead to higher-order terms in $G(X)$. So indeed one may not be able to model $F(X)$ and $G(X)$ independently, provided the Gaussian distribution for transverse fluctuations is assumed.

Let us now consider longitudinal component of pressure gradient increment $\mathcal{P}_{x,u}=\langle\Delta\partial_x p|U,V,r\rangle$, which has a major role in determining the main dynamical aspect of the inverse cascade, i.e., the nonequilibrium energy flux. In contrast to the transversal case, for the longitudinal case both $\langle\Delta\partial_x p|U,r\rangle=\int_{-\infty}^{+\infty}\mathcal{P}_{x,u}P(V|U,r)dV$ and $\langle\Delta\partial_x p|V,r\rangle=\int_{-\infty}^{+\infty}\mathcal{P}_{x,u}P(U|V,r)dU$ are nontrivial. However, the resulting longitudinal PDF equation involves only the $\langle\Delta\partial_x p|U,r\rangle$ and the velocity conditional average $\langle V^2|U,r\rangle$, as one can verify by integrating Eq. (2) over V. Therefore, only the knowledge of these two conditional averages is sufficient to close the longitudinal PDF equation. Again the existence of the velocity conditional average indicates the importance of correlation effects in pure longitudinal statistics. The very existence of a nonequilibrium flux implies that $P(U,r)=P(-U,-r)$, hence the PDF equation would preserve the same invariance, i.e., $\langle V^2|-U,-r\rangle=\langle V^2|U,r\rangle$ and $\langle\Delta\partial_x p|-U,-r\rangle=-\langle\Delta\partial_x p|U,r\rangle$. Scaling invariance of the PDF equation implies scaling invariance of $\mathcal{P}_{x,u}$ and $\langle V^2|U,r\rangle$. Analogous to the transversal case, we assume a local scale-invariant expansion for $\langle\Delta\partial_x p|U,r\rangle$ and $\langle V^2|U,r\rangle$, and we seek for a low-order closure in terms of $Y=U/(\varepsilon r)^{1/3}$. So defining $\langle\Delta\partial_x p|U,r\rangle=[(\varepsilon r)^{2/3}/r]H(Y)$ and $\langle V^2|U,r\rangle=(\varepsilon r)^{2/3}M(Y)$, we propose the following expansion:

$$H(Y)=E\left(Y^2-\frac{3}{5}M(Y)-\frac{6}{5A_{2,0}}Y\right), \tag{8}$$

$$M(Y)=A_{2,0}\left(\frac{5}{3}+\frac{Y}{2A_{2,0}^2}\right). \tag{9}$$

The coefficients of the three terms in the conditional pressure gradient are constrained by homogeneity, isotropy, and incompressibility (i.e., $\overline{YH(Y)}=0$ and $\overline{H(Y)}=0$). We observe that having reduced the expansion of $M(Y)$ at the first order, the only new coefficient is the constant E. In Fig. 2 we show the numerical evaluation of $H(Y)$ and $M(Y)$. From the figure a low-order expansion in terms of Y can be inferred for both these objects. However, concerning $M(Y)$ the result is hardly distinguishable from an almost constant value. From a best fit we found $E=-0.39$ with an error bar around 20%. If the longitudinal fluctuations were purely Gaussian then these models might be considered as a better approximation for $H(Y)$ and $M(Y)$. However, the longitudinal statistics is just nearly Gaussian, indeed the nonzero flux implies a nonzero skewness and to the nonzero odd order structure functions $S_{2n+1,0}(r)=A_{2n+1,0}(\varepsilon r)^{(2n+1)/3}$. Furthermore, a very important observation in Ref. [5] indicates that the hyperskewness of higher orders, i.e., $S_{2n+1,0}/S_{2,0}^{(2n+1)/2}$, increases with order and cannot be considered as a small parameter. So the expectation from any kind of modeling for $H(Y)$ and $M(Y)$ is taking care of these fine details of the distribution. It seems improbable to have access to these fine details within a one-parameter low-order closure or other low-order models. As a quantitative check one can plug the low-order expansion in the longitudinal PDF equation. Then it is straightforward to obtain the following prediction

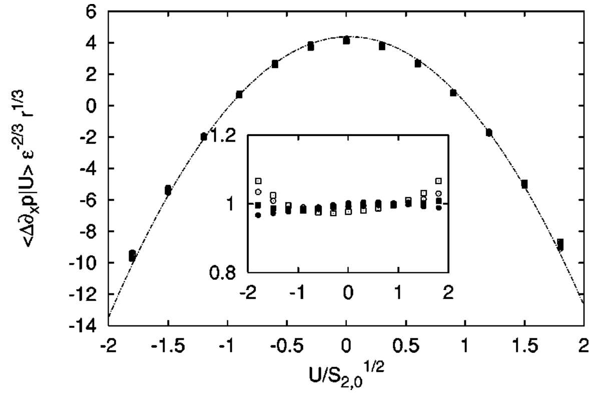

FIG. 2. $\langle\Delta\partial_x p|U,r\rangle$ computed at $r=0.025$ (boxes) and $r=0.037$ (circles). Empty symbols refer to the Gaussian forcing and full ones to the one restricted in a band of wave numbers. The full line is fitted with $E=-0.39$. In the inset we show $\langle V^2|U,r\rangle$ which up to two standard deviations seems to be constant and fluctuates at larger values.

$$A_{2n+1,0}=\frac{2n}{2n\left(E+\frac{1}{3}\right)+\frac{4}{3}}\left\{\left[(2n-1)(2n-3)!!+\frac{(3E+1)(2n-1)!!}{2}\right]A_{2,0}^{n-1}+A_{2,0}\times\left(E+\frac{5}{3}\right)A_{2n-1,0}\right\}. \tag{10}$$

Substituting the numerical value of E we obtain for the hyperskewness $A_{5,0}/A_{2,0}^{5/2}\sim 0.449$ and $A_{7,0}/A_{2,0}^{7/2}\sim 5.674$. Comparing these numbers with the corresponding numerically obtained values, $A_{5,0}/A_{2,0}^{5/2}\sim 0.25$ and $A_{7,0}/A_{2,0}^{7/2}\sim 1.55$, shows a large difference. The fourth- and sixth-order hyperflatnesses calculated from the closure correspondingly are $A_{4,0}/A_{2,0}^{2}\sim 3.29$ and $A_{6,0}/A_{2,0}^{3}\sim 20.03$. Comparing to the Gaussian values the deviations are getting bigger with the order but still the errors are smaller in the even part with respect to the odd part of the statistics. This is an important indication that one has to consider higher-order expansions in order to be consistent with higher-order statistics. Therefore, in spite of the fairly good compatibility between the low-order closure for $H(Y)$ and $M(Y)$ and their direct measurement in two standard deviations, the fine details of the distribution are not recovered by them. This confirms the observation in Ref. [5] that these fine details are buried in the very far tails of the antisymmetric part of longitudinal PDF.

In conclusion, the dynamical role of the pressure gradient and velocity conditional averages in establishing the velocity increment statistics has been highlighted and numerically investigated. The transversal components of the velocity statistics has been found to be Gaussian, in agreement with previous numerical and experimental observations. Low-order expansions for the transversal conditional pressure gradient and $\langle U|V\rangle$, which have been proposed (in a closely related approach) in the context of a self-consistent closure [8], have been found in good agreement with the DNS data up to two standard deviations. Further, we proposed a generalization of the expansion which is order by order consistent with Gaussianity of the transverse statistics. Concerning the longitudinal statistics we found that the low-order closure for the conditional pressure gradient and $\langle V^2|U\rangle$, although in fairly good compatibility with DNS data, is inconsistent with the fine details of the longitudinal PDF, which bear the information of the antisymmetric PDF tail. This indicates that unlike the transverse statistics a complete description of the longitudinal statistics calls for higher-order terms in the expansions [16]. It is worth emphasizing that these modelings are not just a naive fitting: the free parameters are fixed *via* realizability conditions in the PDF equations and have been tested numerically. Let us finally remark that the importance of the conditional averages goes far beyond the assessment of closures for two-dimensional turbulence, the important message is that any theoretical approach to pure longitudinal (transversal) velocity statistics cannot disregard the reciprocal dependence between longitudinal and transversal components. We consider the investigation of such objects also in three-dimensional turbulence to be a necessary step.

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